On the Existence of Periodic Solutions of a Three-Patch Diffusion Predator-Prey System

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We establish a mathematical model for the three-patch diffusion predator-prey system with time delays. The theory of Hopf bifurcation is implemented, choosing the time delay parameter as a bifurcation parameter. We present the condition for the existence of a periodic orbit of the Hopf-type from the positive equilibrium.

Key words: Predator-Prey Model; Time Delay; Diffusion; Hopf Bifurcation; Periodic Solutions.

1. Introduction and Some Notations

One of the first successes of mathematical ecology was the demonstration of periodic population oscillations in a stationary medium. The model created by Volterra for a community in which organisms of one population provide food for those of the other, cleared up the many, at first sight incomprehensible, phenomena of periodic population change, which in no way could be associated with periodic variations of environmental factors (primarily, climatic ones). Similar natural phenomena could be observed in communities with one population parasitizing on the organisms of another species. Communities of such type are usually named predator-prey or host-parasite systems. It is known that time delays have the tendency of producing oscillations or periodic solutions in otherwise nonoscillatory models of single species growths. This is also true for multi-species systems. It would be interesting to know, how the system behaviour is affected when the environmental conditions are impaired (for the predator), the fertility of the prey is enhanced, or some new defence strategies are employed. If originally the system has no nontrivial stable equilibria and produces no oscillations, the impairment first results in damped oscillations. Though stability of the nontrivial equilibrium is preserved, the stability domain is reduced, the predator already fails to regulate the predator population in any domain of the phase plane. Next the oscillations become undamped and a stable limit cycle emerges; these oscillations appear suddenly.

The predator-prey model with or without time delay has been extensively investigated. Many results regarding boundedness, stability, permanence and existence of periodic solutions have been obtained and can be found in some monographs (e. g. [1 – 3]).

The time delay effect or diffusion between patches refer to the dynamics of a predator being related to the predation in the past. Moreover, due to the spatial heterogeneity and unbalanced food resources, the migration phenomenon of biological species can often occur between heterogeneous spatial environments or patches. Mathematicians paid attention to this phenomenon because of its great ecological significance (see [4, 5]).

The present paper deals with a predator-prey model, with time delays, of the form

$$\begin{aligned}
\dot{x}_1 &= x_1(C_1 - x_1 - a_1 y) + \varepsilon(x_2 + x_3 + x_2 x_3 - x_1),\\
\dot{x}_2 &= x_2(C_2 - x_2 - a_2 y) + \varepsilon(x_1 + x_3 + x_1 x_3 - x_2),\\
\dot{x}_3 &= x_3(C_3 - x_3 - a_3 y) + \varepsilon(x_1 + x_2 + x_1 x_2 - x_3),\\
\dot{y} &= y\Big\{ - e + b_1 x_1 + b_2 x_2 + b_3 x_3\\
&\quad + \sum_{i=1}^{3} \beta_i \int_{-\infty}^{t} \alpha_i \exp[-\alpha_i (t-\tau)] x_i(\tau) \mathrm{d}\tau - y \Big\}.
\end{aligned} \tag{1}$$

We first introduced model (1) in [1], where the boundedness and stability of solutions of the system were studied. In the system, we assumed continuous time delays and the prey could diffuse between three patches of a heterogeneous environment with barriers between the patches, but for the predator, the diffusion didn't involve a barrier between the patches. Such a model is known to have a rich ecological background and is conform to more realistic natural situations. From the ecological point of view, time delays and diffusion processes occur simultaneously very often, in almost every true situation.

The densities of predator and prey at three patches in (1), specified below, are rescaled so that the intraspecific coefficients are equal to 1: x_i is the density of prey in the patch i $(i = 1,2,3)$; y is the density of predators; C_i $(i = 1,2,3)$ describes the carrying capacity of the prey in the patch i; e represents the intrinsic death rate of the predator in an environment without any prey involved; a_i, b_i $(i = 1,2,3)$ are the coefficients of instantaneous predation in the patch i; α_i $(i = 1,2,3)$ is the time delay parameter; and ε is the diffusion coefficient between three patches for the prey. In the above parameters, we assume that $\beta_i \geq 0$ $(i = 1,2,3)$ and the remaining parameters are all positive.

2. Stability Analysis

Taking advantage of the Hopf bifurcation theory and choosing the time delay parameter as a bifurcation parameter, we present the condition for the existence of a periodic orbit of the Hopf-type from the positive equilibrium. The methods used here are adopted from Zhujun et al. [6, 7].

We need to introduce three supplementary nonnegative variables:

$$x_4 = \int_{-\infty}^{t} \alpha_1 \exp[-\alpha_1(t-\tau)]x_1(\tau)\mathrm{d}\tau,$$

$$x_5 = \int_{-\infty}^{t} \alpha_2 \exp[-\alpha_2(t-\tau)]x_2(\tau)\mathrm{d}\tau,$$

$$x_6 = \int_{-\infty}^{t} \alpha_3 \exp[-\alpha_3(t-\tau)]x_3(\tau)\mathrm{d}\tau.$$

Then model (1) can be transformed into the following equivalent autonomous differential system:

$$\begin{aligned}
\dot{x}_1 &= x_1(C_1 - x_1 - a_1 y) + \varepsilon(x_2 + x_3 + x_2 x_3 - x_1),\\
\dot{x}_2 &= x_2(C_2 - x_2 - a_2 y) + \varepsilon(x_1 + x_3 + x_1 x_3 - x_2),\\
\dot{x}_3 &= x_3(C_3 - x_3 - a_3 y) + \varepsilon(x_1 + x_2 + x_1 x_2 - x_3),\\
\dot{x}_4 &= \alpha_1 x_1 - \alpha_1 x_4,\\
\dot{x}_5 &= \alpha_2 x_2 - \alpha_2 x_5,\\
\dot{x}_6 &= \alpha_3 x_3 - \alpha_3 x_6,\\
\dot{y} &= y(-e + b_1 x_1 + b_2 x_2 + b_3 x_3\\
&\quad + \beta_1 x_4 + \beta_2 x_5 + \beta_3 x_6 - y).
\end{aligned} \tag{2}$$

Now we give sufficient conditions for system (2) to have a nonzero equilibrium, which is globally asymptotic and stable in R^7_+. It can be shown that $R^7_+ = \{z = (x_1,x_2,x_3,x_4,x_5,x_6,y)^T \in R^7 | z \geq 0\}$ is a positively invariant set with respect to system (2). For the proof see [1].

It is obvious that system (2) can admit three kinds of equilibrium in R^7_+:

$$E_0 = (0,0,0,0,0,0,0),$$

$$E' = (x'_1 > 0, x'_2 > 0, x'_3 > 0,\; x'_4 = x'_1, x'_5 = x'_2, x'_6 = x'_3, 0),$$

$$E^* = (x^*_1 > 0, x^*_2 > 0, x^*_3 > 0,\; x^*_4 = x^*_1, x^*_5 = x^*_2, x^*_6 = x^*_3, y^* > 0).$$

We observe that the positive equilibrium E^* satisfies the system

$$\begin{aligned}
&x^*_1 = x^*_4, \quad x^*_2 = x^*_5, \quad x^*_3 = x^*_6,\\
&x^*_1(C_1 - x^*_1 - a_1 y^*_1) + \varepsilon(x^*_2 + x^*_3 + x^*_2 x^*_3 - x^*_1) = 0,\\
&x^*_2(C_2 - x^*_2 - a_2 y^*_2) + \varepsilon(x^*_1 + x^*_3 + x^*_1 x^*_3 - x^*_2) = 0,\\
&x^*_3(C_3 - x^*_3 - a_3 y^*_3) + \varepsilon(x^*_1 + x^*_2 + x^*_1 x^*_2 - x^*_3) = 0,\\
&y^*\big[-e + (b_1+\beta_1)x^*_1 + (b_2+\beta_2)x^*_2\\
&\quad + (b_3+\beta_3)x^*_3 - y^*\big] = 0.
\end{aligned}$$

The existence of the positive equilibrium E^* in (2) can be obtained as follows:

Let us introduce the auxiliary system

$$\begin{aligned}
\dot{x}_1 &= x_1(C_1 - x_1 - a_1 y) + \varepsilon(x_2 + x_3 + x_2 x_3 - x_1),\\
\dot{x}_2 &= x_2(C_2 - x_2 - a_2 y) + \varepsilon(x_1 + x_3 + x_1 x_3 - x_2),\\
\dot{x}_3 &= x_3(C_3 - x_3 - a_3 y) + \varepsilon(x_1 + x_2 + x_1 x_2 - x_3),\\
\dot{y} &= y\big[-e + (b_1+\beta_1)x_1\\
&\quad + (b_2+\beta_2)x_2 + (b_3+\beta_3)x_3 - y\big].
\end{aligned} \tag{3}$$

It can be shown that, if (x^*_1, x^*_2, x^*_3, y) is a positive equilibrium of (1), $(x^*_1, x^*_2, x^*_3, x^*_4 = x^*_1, x^*_5 = x^*_2, x^*_6 = x^*_3, y^*)$ is positive equilibrium of system (2).

Therefore, it is necessary to discuss the existence of the positive equilibrium of system (3).

Applying some results provided in [8, 9], the following lemma can be proved.

Lemma 1. In system (3), there exists a unique equilibrium of the form $E' = E'(\varepsilon) = (x_1'(\varepsilon), x_2'(\varepsilon), x_3'(\varepsilon), 0)$, where $x_1'(\varepsilon) > 0, i = 1,2,3$. Now, we define

$$d = d(\varepsilon) = -e + (b_1+\beta_1)x_1'(\varepsilon) + (b_2+\beta_2)x_2'(\varepsilon) + (b_3+\beta_3)x_3'(\varepsilon).$$

If $d > 0$, then system (3) has a positive equilibrium E^*.

Proof. The proof of this lemma is cited in [1].

3. Main Results

Considering that the time delay effect is involved with the same prey species in three patches, we may suppose that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. Now, by choosing α as a bifurcation parameter, let us consider the conditions for the existence of the periodic orbits of the Hopf-type from the positive equilibrium E^* in system (2).

The Jacobian matrix of (2) at E^* is expressed as

$$J = \begin{bmatrix} C_1 - 2x_1^* - a_1y^* - \varepsilon & \varepsilon + \varepsilon x_3^* & \varepsilon + \varepsilon x_2^* & 0 & 0 & 0 & -a_1x_1^* \\ \varepsilon + \varepsilon x_3^* & C_2 - 2x_2^* - a_2y^* - \varepsilon & \varepsilon + \varepsilon x_1^* & 0 & 0 & 0 & -a_2x_2^* \\ \varepsilon + \varepsilon x_2^* & \varepsilon + \varepsilon x_1^* & C_3 - 2x_3^* - a_1y^* - \varepsilon & 0 & 0 & 0 & -a_3x_3^* \\ \alpha_1 & 0 & 0 & -\alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & -\alpha_3 & 0 \\ b_1y^* & b_2y^* & b_3y^* & \beta_1y^* & \beta_2y^* & \beta_3y^* & A_4 \end{bmatrix}, \quad (4)$$

where $A_4 = -e + b_1x_1^* + b_2x_2^* + b_3x_3^* + \beta_1x_4^* + \beta_2x_5^* + \beta_3x_6^* - 2y^*$. Let $A_1 = C_1 - 2x_1^* - a_1y^* - \varepsilon,\ A_2 = C_2 - 2x_2^* - a_2y^* - \varepsilon,\ A_3 = C_3 - 2x_3^* - a_3y^* - \varepsilon$. Then

$$|J - \lambda I| = \begin{bmatrix} A_1 - \lambda & \varepsilon + \varepsilon x_3^* & \varepsilon + \varepsilon x_2^* & 0 & 0 & 0 & -a_1x_1^* \\ \varepsilon + \varepsilon x_3^* & A_2 - \lambda & \varepsilon + \varepsilon x_1^* & 0 & 0 & 0 & -a_2x_2^* \\ \varepsilon + \varepsilon x_2^* & \varepsilon + \varepsilon x_1^* & A_3 - \lambda & 0 & 0 & 0 & -a_3x_3^* \\ \alpha_1 & 0 & 0 & -\alpha_1 - \lambda & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\alpha_2 - \lambda & 0 & 0 \\ 0 & 0 & -\alpha_3 & 0 & 0 & -\alpha_3 - \lambda & 0 \\ b_1y^* & b_2y^* & b_3y^* & \beta_1y^* & \beta_2y^* & \beta_3y^* & A_4 - \lambda \end{bmatrix}.$$

By computation, the corresponding characteristic equation of the eigenvalues for (4) can be found in the following form:

$$\begin{aligned} p(\lambda) &= |J - \lambda I| \\ &= \lambda^7 + I_1(\alpha)\lambda^6 + I_2(\alpha)\lambda^5 \\ &\quad + I_3(\alpha)\lambda^4 + I_4(\alpha)\lambda^3 + I_5(\alpha)\lambda^2 \\ &\quad + I_6(\alpha)\lambda^1 + I_7(\alpha) = 0, \end{aligned} \quad (5)$$

where

$$\begin{aligned} I_1(\alpha) &= \alpha_1 + \alpha_2 + \alpha_3 - A_1 - A_2 - A_3 - A_4, \\ I_2(\alpha) &= -\alpha_3A_1 - \alpha_2A_2 - \alpha_3A_4 - \alpha_2A_3 + A_1A_3 \\ &\quad - 2\varepsilon^2x_3^* + b_1y^*a_1x_1^* + a_3x_3^*b_3y^* + b_2y^*a_2x_2^* \\ &\quad + \alpha_2\alpha_1 + \alpha_2\alpha_3 + 3\varepsilon^2 + \alpha_1\alpha_3 - \alpha_3A_2 \\ &\quad + A_2A_3 - \alpha_2A_1 + A_2A_4 - \alpha_3A_3 - \alpha_1A_1 \\ &\quad - 2\varepsilon^2x_1^* - 2\varepsilon^2x_2^* - \alpha_1A_4 - \varepsilon^2x_3^{*2} - \alpha_2A_4 \\ &\quad + A_3A_4 - \varepsilon^2x_1^{*2} + A_1A_2 - \varepsilon^2x_2^{*2} - \alpha_1A_3 \\ &\quad + A_1A_4 - \alpha_1A_2. \end{aligned}$$

We use the software Maple 9.5 Software in our computations and, for convenience, we do not introduce the remaining coefficient functions.

Lemma 2. If the conditions

(H1) $I_i(\alpha) > 0, \quad i = 1,2,\ldots,7,$
$\Delta_2 > 0, \quad \Delta_4 > 0,$

(H2) $\Delta = 0$

are satisfied, then the characteristic equation (5) has a pair of purely imaginary roots, and the remaining roots have negative real parts.

Proof. It is known from [10] that the Hurwitz determinant

$$\Delta = \begin{vmatrix} I_1 & I_3 & I_5 & I_7 & 0 & 0 \\ 1 & I_2 & I_4 & I_6 & 0 & 0 \\ 0 & I_1 & I_3 & I_5 & I_7 & 0 \\ 0 & 1 & I_2 & I_4 & I_6 & 0 \\ 0 & 0 & I_1 & I_3 & I_5 & I_7 \\ 0 & 0 & 1 & I_2 & I_4 & I_6 \end{vmatrix} = 0,$$

if and only if (5) has a pair of opposite roots, ω and $-\omega$, and satisfies the following equations:

$$p_1(\lambda) = \lambda^7 + I_2\lambda^5 + I_4\lambda^3 + I_6\lambda = 0,$$

$$p_2(\lambda) = I_1\lambda^6 + I_3\lambda^4 + I_5\lambda^2 + I_7 = 0.$$

Thus, $I_1\omega^6 + I_3\omega^4 + I_5\omega^2 + I_7 = 0$ by (H1). Since $I_1, I_3, I_5, I_7 > 0$, ω^2 is a negative real number or complex number. In the latter case, let $\omega = a + \mathrm{i}b, a \neq 0, b \neq 0$. Then $P(\lambda)$ can be resolved as follows:

$$P(\lambda) = (\lambda^2 - \omega^2) \cdot (\lambda^5 + b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda + b5).$$

By comparing the coefficients with (4), we obtain

$$b_1 = I_1, \quad b_2 = I_2 + \omega^2, \quad b_3 = I_3 + b_1\omega^2,$$
$$b_4 = I_4 + b_2\omega^2, \quad b_5 = I_5 + b_3\omega^2,$$

$$I_6 = -b_4\omega^2, \quad I_7 = -b_5\omega^2.$$

Moreover, $a - b\mathrm{i}$ and $-a + b\mathrm{i}$ are also the roots of

$$P_3(\lambda) = \lambda^5 + b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda + b_5 = 0.$$

Therefore, we have

$$\Delta_b = \begin{vmatrix} b_1 & b_3 & b_5 & 0 \\ 1 & b_2 & b_4 & 0 \\ 0 & b_1 & b_3 & b_5 \\ 0 & 1 & b_2 & b_4 \end{vmatrix}$$
$$= b_1b_2b_3b_4 - b_1b_5b_2^2 - b_1^2b_4^2 + 2b_1b_4b_5 - b_3^2b_4 + b_2b_3b_5 - b_5^2 = 0,$$

which leads to

$$\Delta_4 = \begin{vmatrix} I_1 & I_3 & I_5 & I_7 \\ 1 & I_2 & I_4 & I_6 \\ 0 & I_1 & I_3 & I_5 \\ 0 & 1 & I_2 & I_4 \end{vmatrix}$$
$$= b_1(b_2 - \omega^2)(b_3 - \omega^2 b_1)(b_4 - \omega^2 b_2) - b_1(b_2 - \omega^2)(b_5 - \omega^2 b_3) - b_1^2(b_4 - \omega^2 b_2) + 2b_1(b_4 - \omega^2 b_2)(b_5 - \omega^2 b_3) + b_1^2(b_2 - \omega^2)(-\omega^2 b_4) - b_1(b_3 - \omega^2 b_1)(-\omega^2 b_4) - (b_3 - \omega^2 b_1)^2(b_4 - \omega^2 b_2) + (b_2 - \omega^2)(b_3 - \omega^2 b_1)(b_5 - \omega^2 b_3) - (b_5 - \omega^2 b_3)^2 - b_1(b_2 - \omega^2)(-\omega^2 b_5) + (b_3 - \omega^2 b_1)(-\omega^2 b_5)$$
$$= b_1b_2b_3b_4 + 2b_1b_4b_5 + b_2b_3b_5 - b_1b_2^2b_5 - b_1^2b_4^2 - b_3^2b_4 - b_5^2 = 0.$$

This contradicts assumption (H1), and hence ω^2 is a negative real number, which implies that (4) has a pair of purely imaginary roots $\pm\mathrm{i}\omega$.

Since ω satisfies $P_1(\lambda) = 0$ and $P_2(\lambda) = 0$, we have

$$\omega^6 + I_2\omega^4 + I_4\omega^2 + I_6 = 0 \quad \text{and}$$
$$I_1\omega^6 + I_3\omega^4 + I_5\omega^2 + I_7 = 0.$$

It can be obtained that

$$\omega^4 = \frac{I_5 - I_1I_4}{I_1I_2 - I_3} + \frac{I_7 - I_1I_6}{I_1I_2 - I_3}.$$

By (H1), $\Delta_4 > 0$; then $b_1b_2b_3b_4 + 2b_1b_4b_5 + b_2b_3b_5 - b_1b_2^2b_5 - b_1^2b_4^2 - b_3^2b_4 - b_5^2 > 0$.

By the well-known Routh-Hurwitz stability condition, the roots of the equation

$$P_3(\lambda) = \lambda^5 + b_1\lambda^4 + b_2\lambda^3 + b_3\lambda^2 + b_4\lambda + b_5 = 0$$

all have negative real parts. This completes the proof.

The following theorem dominates our main result concerning the existence of periodic orbits in system (2).

Theorem. For system (2), we suppose that the coefficients of the characteristic equation (4) are all functions of the parameter α, if $\alpha = \alpha^* > 0$, such that

(H3) $I_i(\alpha^*) > 0, \quad i = 1, 2, \ldots, 7, \quad \Delta_2(\alpha^*) > 0,$

(H4) $\Delta(\alpha^*) = 0$,

(H5) Denote that

$$\Delta'(\alpha) = [(2I_2^2 - 14I_2^2 I_4 + 8I_2 I_6 + 10I_4^2)\omega^2 + (2I_2^2 I_4 - 2I_2^2 I_6 - 12I_2 I_4^2 + 16I_4 I_6)\omega^2]\frac{\mathrm{d}I_1}{\mathrm{d}\alpha} + (2I_2^3 I_4 - 12I_2 I_4 I_6 + 6I_6^2)\frac{\mathrm{d}I_1}{\mathrm{d}\alpha} + [(6I_1 I_2^3 - 6I_1 I_2 I_4 + 6I_1^2 I_4 - 6I_1 I_6 - 4I_2^2 I_3 + 4I_3 I_4 - 2I_2 I_5)\omega^2 + (6I_1 I_4^2 - 6I_1 I_2^2 I_4 - 4I_2 I_3 I_4 + 4I_3 I_6 - 2I_4 I_5)\omega^2 + (6I_1 I_2 I_6 - 6I_1 I_2^2 I_6 + 6I_1 I_4 I_6 - 4I_2 I_3 I_6 - 2I_5 I_6)]\frac{\mathrm{d}I_2}{\mathrm{d}\alpha} + [(12I_2 I_4 - 2I_2^3 - 6I_6)\omega^4 + (2I_2 I_6 - 2I_2^2 I_4 + 10I_4^2)\omega^2 + (10I_4 I_6 - 2I_2^2 I_6)]\frac{\mathrm{d}I_3}{\mathrm{d}\alpha} - [(6I_1 I_2^2 - 6I_1 I_4 + 4I_2 I_4 + 2I_5)\omega^4 + (6I_1 I_2 I_4 - 6I_1 I_6 + 4I_3 I_4)\omega^2 + (6I_1 I_2 I_6 + 4I_3 I_6)]\frac{\mathrm{d}I_4}{\mathrm{d}\alpha} - [(2I_2^2 - 7I_4)\omega^4 + (2I_2 I_4 - 6I_6 - 3I_4)\omega^2 - 2I_2 I_6]\frac{\mathrm{d}I_5}{\mathrm{d}\alpha} + [(-6I_1 I_2 - 4I_3)\omega^2 + (-6I_1 I_4 + 2I_5)\omega^2 - 6I_1 I_6]\frac{\mathrm{d}I_6}{\mathrm{d}\alpha} + [-2I_2\omega^4 - 10I_4\omega^2 - 6I_6]\frac{\mathrm{d}I_7}{\mathrm{d}\alpha},$$

$$\Delta'(\alpha)|_{\alpha=\alpha^*} \neq 0,$$

holds, then a periodic orbit of the Hopf-type bifurcation from the positive equilibrium E^* occurs as the value of α passes through α^*.

Proof. If there exists an $\alpha = \alpha^* > 0$, such that the conditions (H3) and (H4) hold, then, by Lemma 2, the characteristic equation (4) at the equilibrium E^* has a pair of purely imaginary roots, and the remaining roots have negative real parts. According to the Hopf bifurcation theorem (see [11]), it is necessary to verify that $\mathrm{Re}\{\mathrm{d}\lambda/\mathrm{d}\alpha\} \neq 0$ at $\alpha = \alpha^*$.

From (4), by directly calculating $\mathrm{d}\lambda/\mathrm{d}\alpha$, we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\alpha} = \left[\frac{\mathrm{d}I_1}{\mathrm{d}\alpha}\lambda^6 + \frac{\mathrm{d}I_2}{\mathrm{d}\alpha}\lambda^5 + \frac{\mathrm{d}I_3}{\mathrm{d}\alpha}\lambda^4 + \frac{\mathrm{d}I_4}{\mathrm{d}\alpha}\lambda^3 + \frac{\mathrm{d}I_5}{\mathrm{d}\alpha}\lambda^2 + \frac{\mathrm{d}I_6}{\mathrm{d}\alpha}\lambda + \frac{\mathrm{d}I_7}{\mathrm{d}\alpha}\right]\left[7\lambda^6 + 6I_1\lambda^5 + 5I_2\lambda^4 + 4I_3\lambda^3 + 3I_4\lambda^2 + 2I_5\lambda + I_6\right]^{-1}.$$

Therefore,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\alpha}\Big|_{\lambda=\mathrm{i}\omega} = \left[\left(\frac{\mathrm{d}I_1}{\mathrm{d}\alpha}\omega^6 + \frac{\mathrm{d}I_3}{\mathrm{d}\alpha}\omega^4 + \frac{\mathrm{d}I_5}{\mathrm{d}\alpha}\omega^2 + \frac{\mathrm{d}I_7}{\mathrm{d}\alpha}\right) + \left(\frac{\mathrm{d}I_2}{\mathrm{d}\alpha}\omega^5 + \frac{\mathrm{d}I_4}{\mathrm{d}\alpha}\omega^3 + \frac{\mathrm{d}I_6}{\mathrm{d}\alpha}\omega\right)\mathrm{i}\right]\left[(7\omega^6 + 5I_2\omega^4 + 3I_4\omega^2 + I_6) + (6I_1\omega^5 + 4I_3\omega^3 + 2I_5\omega)\mathrm{i}\right]^{-1}.$$

Thus we obtain

$$\mathrm{Re}\left\{\frac{\mathrm{d}\lambda}{\mathrm{d}\alpha}\right\}\Big|_{\alpha=\alpha^*} = \frac{\Delta'(\alpha)}{Q(\alpha)},$$

where

$$\Delta'(\alpha) = \left(\frac{\mathrm{d}I_1}{\mathrm{d}\alpha}\omega^6 + \frac{\mathrm{d}I_3}{\mathrm{d}\alpha}\omega^4 - \frac{\mathrm{d}I_5}{\mathrm{d}\alpha}\omega^2 + \frac{\mathrm{d}I_7}{\mathrm{d}\alpha}\right)\cdot(7\omega^6 + 5I_2\omega^4 - 3I_4\omega^2 + I_6) + \left(\frac{\mathrm{d}I_2}{\mathrm{d}\alpha}\omega^5 - \frac{\mathrm{d}I_4}{\mathrm{d}\alpha}\omega^3 + \frac{\mathrm{d}I_6}{\mathrm{d}\alpha}\right)\cdot(6I_1\omega^5 - 4I_3\omega^3 + 2I_5\omega),$$

$$Q(\alpha) = (7\omega^6 + 5I_2\omega^4 - 3I_4\omega^2 + I_6)^2 + (6I_1\omega^5 - 4I_3\omega^3 + 2I_5\omega)^2.$$

From the proof of Lemma 2, we see that

$$\omega^6 + I_2\omega^4 + I_4\omega^2 + I_6 = 0,$$

$$\omega^4 = \frac{I_5 - I_1 I_4}{I_1 I_2 - I_3} + \frac{I_7 - I_1 I_6}{I_1 I_2 - I_3}.$$

Hence, after simplifying, we get

$$\Delta'(\alpha) = [(2I_2^2 - 14I_2^2 I_4 + 8I_2 I_6 + 10I_4^2)\omega^4 + (2I_2^2 I_4 - 2I_2^2 I_6 - 12I_2 I_4^2 + 16I_4 I_6)\omega^2]\frac{\mathrm{d}I_1}{\mathrm{d}\alpha} + (2I_2^3 I_4 - 12I_2 I_4 I_6 + 6I_6^2)\frac{\mathrm{d}I_1}{\mathrm{d}\alpha} + [(6I_1 I_2^3 - 6I_1 I_2 I_4 + 6I_1^2 I_4 - 6I_1 I_6 - 4I_2^2 I_3 + 4I_3 I_4 - 2I_2 I_5)\omega^2 + (6I_1 I_4^2 - 6I_1 I_2^2 I_4 - 4I_2 I_3 I_4 + 4I_3 I_6 - 2I_4 I_5)\omega^2 + (6I_1 I_2 I_6 - 6I_1 I_2^2 I_6 + 6I_1 I_4 I_6$$

$$
\begin{aligned}
&- 4I_2I_3I_6 - 2I_5I_6)]\frac{\mathrm{d}I_2}{\mathrm{d}\alpha} \\
&+ [(12I_2I_4 - 2I_2^3 - 6I_6)\omega^4 \\
&+ (2I_2I_6 - 2I_2^2I_4 + 10I_4^2)\omega^2 \\
&+ (10I_4I_6 - 2I_2^2I_6)]\frac{\mathrm{d}I_3}{\mathrm{d}\alpha} \\
&- [(6I_1I_2^2 - 6I_1I_4 + 4I_2I_4 + 2I_5)\omega^4 \\
&+ (6I_1I_2I_4 - 6I_1I_6 + 4I_3I_4)\omega^2 \\
&+ (6I_1I_2I_6 + 4I_3I_6)]\frac{\mathrm{d}I_4}{\mathrm{d}\alpha} \\
&- [(2I_2^2 - 7I_4)\omega^4 \\
&+ (2I_2I_4 - 6I_6 - 3I_4)\omega^2 - 2I_2I_6]\frac{\mathrm{d}I_5}{\mathrm{d}\alpha} \\
&+ [(-6I_1I_2 - 4I_3)\omega^2 \\
&+ (-6I_1I_4 + 2I_5)\omega^2 - 6I_1I_6]\frac{\mathrm{d}I_6}{\mathrm{d}\alpha} \\
&+ [-2I_2\omega^4 - 10I_4\omega^2 - 6I_6]\frac{\mathrm{d}I_7}{\mathrm{d}\alpha}.
\end{aligned}
$$

By (H5), we have

$$\mathrm{Re}\left\{\frac{\mathrm{d}\lambda}{\mathrm{d}\alpha}\right\}\Big|_{\alpha=\alpha^*} \neq 0.$$

The proof of the theorem is now completed.

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